A Random Mapping Statistics and a Related Identity

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Investigating elementary properties of random selections without displacement from a finite, discrete set, one derives identities which can be cast either as a finite probability distribution or as a decomposition of n^n. They are reported here, together with notes on the original object of this study.

Introduction

This tiny essay started as a reflection on how soon should a random numbers generator repeat a value[1]. This is a classical problem in the art of random number generators [1-4], but here we will pursue a point of view which is a bit off-stream.

Terminology: Let us start with a random mapping $R(i) \rightarrow S$ of natural numbers i into a set S of n elements $\{s_1, s_2, ..., s_n\}$. The result of an application $R(i)$, henceforth called a "shot", is one of the elements of S, with all of them having the same probability to be "hit". We will be interested in how many times various elements of S were hit after K shots.

Premises: The only purpose of the argument i of $R(i)$ is to label the i-*th* shot. Upon every shot an element of S gets certainly hit[2], but there is no way to know which one, and the outcomes of consecutive shots are independent of each other. Hitting an element does not remove it from the set S. We actually implicitly assume that each element "remembers" how many times it was hit.

There are many approximate practical models of this type of processes, such as:

- Running a discrete random numbers generator (well, deep down, all RNG's are discrete).
- Bombing an area that was partitioned into a discrete number of equal squares.
- Bombardment of a celestial body by meteorites (divide the body surface into finite elements).
- Winning a raffle round [5] in a group of habitual players (each game representing one shot).

Since the *probability* of hitting any particular element is known and equal to 1/n, the most obvious queries that arise in this context are of this kind[3]:

1. How many shots does it take for the probability of hitting a particular element to exceed 1/2 ?
2. How many shots does it take, on the average, for half of the elements to get hit?
3. How many shots it takes before the probability of having hit all elements reaches a certain level.
4. How many shots is it likely to take before a *particular* element will be hit twice?
5. How many shots is it likely to take before *some* element will be hit twice?

Most of these are easy questions leading to the binomial statistics [6] with probability $p = 1/n$.
After K shots there will be elements of S that were hit j times, with j = 0, 1, 2, ..., K.
The probabilities $a(K,j)$ of each such outcome are of course the terms in the expansion of $(p+(1-p))^K$:

$$(p+(1-p))^K = \sum_{j=0,K} a(K,j) = 1, \text{ where } a(K,j) = ((n-1)^j/n^K)C(K,j), \qquad (1)$$

C(k,l) being the usual binomial coefficient [7].

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[1] Which does not mean that the generator will start repeating the whole sequence, unless it is really primitive.
[2] In processes where "shots" may fail to hit any element, one can resort to the derived conditional variety whereby a shot is counted "valid" only after it has resulted in a hit.
[3] Both lists are illustrative, but not at all exhaustive.

Note that in this context K is unlimited. Even if it were incomparably larger than n, for example, there might still be elements which did not get hit at all (lucky survivors) and, vice versa, it might potentially happen that every shot would hit the same element (a bad-luck attractor). All this is interesting but conventional and we will not pursue it further in this Note. Let me only point out, for reference purposes, that putting K = n in (1), one obtains the following trivial decomposition of n^n:

$$\sum_{j=0,n} (n-1)^j C(n,j) = n^n. \tag{2}$$

What will interest us more are identities that come up when these statistical considerations are combined with the pigeonhole principle [8,9].

A reflection on multiple hits

If, after i shots, no element was hit twice, what is the probability that the next, (i+1)-*st*, shot will also hit an element which was not yet hit. Since there are exactly (n-i) such unscathed elements, the answer is (n-i)/n. The statement that *prior to the k-th shot, no element was hit twice* is equivalent to saying that no element was hit twice in any of the shots up to *k-th*. The probability of this happening is therefore

$$\prod_{i=0,k-1}(n-i)/n = (1/n^k)[n!/(n-k)!] = (k!/n^k)C(n,k), \tag{3}$$

Note that the shot index i starts formally at 0 (which is inconsequential since the 0-*th* factor is 1) and stops at (k-1), just prior to k.

Complementing (1), one obtains the *cumulative distribution function* [10] for the event of hitting <u>some</u> element more than once in the k shots:

$$D(n,k) = 1 - (k!/n^k)C(n,k). \tag{4}$$

The k in this formula ranges from k=0 with D(n,0) = 0 (no shot, no hit), through 1 with D(n,1)=1 (one shot cannot produce multiple hits), up to k=n with D(n,n) = 1 - n!/n^n.

The validity of D(n,k) can be extended to k=n+1 by setting D(n, n+1)=1. The rationale behind this is provided by the **pigeonhole principle** [8]: if the n elements of S get hit a total of (n+1) times, at least one element *must* be hit more than once, so that the probability of that event must be 1.

The corresponding *probability mass function* [11] that <u>some</u> element will be hit twice in the shot number (j+1), but not before, is then

$$P(n, j) = D(n, j+1) - D(n, j) = (j/n)(j!/n^j)C(n, j), \tag{5}$$

for j = 1 through n. Equations (4) and (5) define a discrete probability distribution on the finite set of j $\in$ {1,2,...,n} shots which looks rather novel, since I could not find a category [12] where to put it.

To understood this distribution properly consider, as a model, a random number generator (RNG) which selects randomly one element out of n. Let us run the RNG until it selects an element which was already hit before, an event which will certainly happen in at most (n+1) shots. Let's say it happened in shot (j+1), i.e., the last shot which did not lead to any repetition was the j-th. Let us consider this process as a single "run" of the RNG, with the number j, 1 ≤ j ≤n, being its outcome. Then the probability density (5) describes the distribution of the values of j *when one repeats the runs* for an unlimited number of times.

It should be now clear why, for n>2, P(n,j) has a maximum. When n is large, the runs when the first double-hit occurred very early (after just a few shots) are bound to be very rare. Likewise, runs where after close to n shots there was still no double hit will be also extremely rare (it is unlikely that almost every element would get hit, but none more than once).

The location of the maximum is best estimated by writing $P(n, j)$ in a recursive form, $P(n, j+1) = f.P(n, j)$, where $f = [(j+1)/j][(n-j)/n]$. Clearly, the factor f is decreasing; it starts at $2(n-1)/n$ which, for $n>2$, is greater than 1, and decreases towards 0. The maximum of $P(n,j)$ therefore occurs just before f crosses 1. Setting $f = 1$ leads to a quadratic equation for j whose solution is

$$j_{max} = \left[\sqrt{4n + 1} - 1\right]/2. \tag{6}$$

The value of j_{max} equals approximately $\sqrt{n}$, with a discrepancy smaller than 1 for any $n > 2$.

One of the consequences of this is that, for example, a "correctly random" RNG should lead to a duplicate hit in about $\sqrt{n}$ shots (which turns out to be quite early). The usual efforts to find RNG's with "as long cycles as possible" are therefore unnatural and cause deviations from randomness.

An interesting identity

The distribution (5) certainly merits to be studied in more detail but that will take another essay. For now, let us just notice a direct corollary which looks quite interesting on its own:

Thanks to the pigeonhole principle, $P(n, j)$ is a true probability density on a discrete set of n elements (shots, not those of S), and the sum of all $P(n, j)$, for $j = 1$ to n, must be 1. Hence

$$\sum_{j=1,n} (j/n)(j!/n^j)\, C(n, j) = 1. \tag{7}$$

Using simple manipulations, this becomes another integer decomposition of n^n,

$$\sum_{j=0,n} (n^{n-j-1})(j)(j!)\, C(n, j) = n^n. \tag{8}$$

Alternatively, it can be also written as an intriguing decomposition of unity:

$$\sum_{j=1}^{n} \frac{j}{n} \prod_{k=1}^{j-1}\left(1 - \frac{k}{n}\right) \equiv \sum_{j=1}^{n}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{j-1}{n}\right)\frac{j}{n} = 1 \tag{9}$$

Note: the first term in (9), for $j = 1$, is simply $1/n$, compatibly with the convention that a product over zero terms ($k = 1$ to 0) evaluates to 1.

Let us compare the n^n expansions (2) and (8). Both have the form

$$\sum_{j=0,n} w(n,j)C(n, j) = n^n, \tag{10}$$

but the 'weight' coefficients $w(n, j)$ are quite different:

in (2), $\quad w(n, j) = (n-1)^j = b(n, j)$ (11)
in (8), $\quad w(n, j) = (n^{n-j-1})(j)(j!) = c(n, j)$ (12)

A numeric example of both for $n= 7$, $j = 0,1,2,...,7$,

$b(7,j) = \{1, \quad 6, \quad 36, \quad 216, 1296, \quad 7776, 46656, 279936\},$
$c(7,j) = \{0, 16807, 9604, 6174, \quad 4704, 4200, \quad 4320, \quad 5040\},$

reveals, apart from the first element, a much more 'balanced' distribution of the values for $c(n,j)$ with even a flat minimum in the upper half. A consequence of this is that the values $w(n,j)C(n,j)$ are in identity (8) much more peaked around the central values of j, rather than close to the end as in identity (2):

$b(7,j)C(7,j) = \{1, \quad 42, \quad 756, \quad 7560, \quad 45360, 163296, \underline{326592}, 279936\},$
$c(7,j)C(7,j) = \{0, 117649, 201684, \underline{216090}, 164640, \quad 88200, 30240, \quad 5040\}.$

Conclusion

Analyzing the process of random statistical selection (or 'shooting') of elements in a finite set typical, for example, of random number generators, we have derived a statistics on the shot number when the first multiple hit occurs. This appears to be a novel type of a discrete distribution function.

It also leads to an interesting identity which can be cast in two ways: either as an integer decomposition of the number n^n in terms of weighed binomial coefficients $C(n,j)$, or as a rational decomposition of unity. The triangular matrices of the n^n decomposition (both the integer 'weights' and the full terms) were registered[4] in OEIS [13] (numbers A243202 and A243203).

The analysis also leads to an insight regarding discrete random number generators on sets with n elements. It turns out that trying forcibly to make an RNG exhibit a "cycle" much longer than about $\sqrt{n}$ is statistically unnatural. Rather, a single-value repetition after about $\sqrt{n}$ steps should be allowed, but avoiding, obviously, a complete repetition of the subsequent sequence.

References and links

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[4]   Press W.H., Teukolsky S.A., Vetterling W.T., Flannery B.P., *Numerical Recipes in C. The Art of Scientific Computing*, Cambridge University Press 1992, Section 7. ISBN 978-0521431088. Partially available online.

[5]   Wikipedia, Raffle.

[6]   Wikipedia, Binomial distribution.

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[8]   Wikipedia, Pigeonhole principle.

[9]   Grimaldi R.P., *Discrete and Combinatorial Mathematics: An Applied Introduction*, 5th edition, Pearson 2003. ISBN 978-0201726343.

[10]   Wikipedia, Cumulative distribution function.9

[11]   Wikipedia, Probability mass function.

[12]   Wikipedia, List of probability distributions.

[13]   OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences.

History of this document

1 June 2014: Assigned a DOI (10.3247/SL5Math14.003) and uploaded online.

2 December 2018: Minor edits for better readability, removal of a typo (RND instead of RNG) in the Introduction, and addition of the Note after equation (9).

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[4] While registering the sequence of the full terms, it turned out that it is not quite novel; it arises also in a different context and was already registered in A066324, but with the starting zero terms omitted. Conceptually, considering the summations over the binomial coefficients $C(n, j)$ I believe that maintaining the $j=0$ terms is preferable, even if they are identically zero. At least in the current context (the first shot, being single, cannot produce a double-hit).