Taylor Traitor Functions

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Semetimes it is not easy to make students (and non-mathematicians) grasp some of the notrivial characteristics of the Taylor power series. In particular, it occasionally comes as a surprise that a function f(x) may, at a given point of its domain, possess derivatives of all orders, give rise to a convergent formal Taylor series, and yet the limit of that series need not at all coincide with f(x). Since a simple example is worth tens of pages, the first part of this educational Note discussed one such example (a well.known case). The Note then proceeds with a more detailed analysis of this special case and its analogues.

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Introduction

Let f(x) be a real function of real variable, ***I*** = (a,b) an interval of the set of real numbers ***R***, and let y be a fixed element of ***I***. We will assume that the following conditions are satisfied:

(a) f(x) has in ***I*** derivatives $f^{(k)}(x)$ of all orders
(b) the following **formal** Taylor series [1,2] **converges** everywhere in ***I*** to a function $u_y(x)$.

(1) $$\sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!}(x-y)^k = u_y(x).$$

At this point, many students try and **identify** $u_y(x)$ with f(x), forgetting that such a step requires an **independent** proof of the convergence to zero of the remainder [3,4] term $R_{n,y}(x)$ in the complete Taylor formula

(2) $$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(y)}{k!}(x-y)^k + R_{n,y}(x).$$

and that *such a convergence is by no means guaranteed* (see Appendix A for more details).

Among all functions which satisfy the conditions (a) and (b), we shall call **Taylor friendly** at x = y those for which $u_y(x) = f(x)$ for every x in ***I***, while those for which $u_y(x) \neq f(x)$ for some x in ***I*** will be called **Taylor traitors**. In the latter case, the discrepancy will often occur at all, or almost all, values of x other than the fixed point y.

Example of a Taylor traitor

A representative example of a Taylor traitor is the function[1]

(3) $$F(x) = \exp(-1/x^2)$$

in the vicinity of x = 0 (see Fig 1).

It is easy to verify by elementary methods that F(x) satisfies condition (a). Moreover, it is shown in Appendix B that $F^{(k)}(0) = 0$ for any k =0, 1, 2, 3, ... This means that *all terms of the formal Taylor series of* F(x) *at* x = 0 *are zero*. Consequently, the series not only exists, it also certainly converges and its limit is everywhere 0 which, however, is **manifestly different** from F(x) !!!

The only way to reconcile this fact with the Taylor formula theorem is to admit/realize that in the case of F(x) *the remainder is equal to the function itself* - a fact which may look unusual but which is not at all incompatible with the theorem itself!

The problems with the theorem are that (i) it does not guarantee the convergence of the formal series and (ii) it does not guarantee the convergence of the remainder to zero. Actually, in general, it does not guarantee that the remainder converges at all.

Can the situation be even worse than that?

Well, it can! Let the function f(x) be Taylor friendly at y and let T(x) be any Taylor traitor at x = 0, such as the F(x) of Eq.(3). Then any function g(x) = f(x)+cT(x-y), where c is a non-zero constant is necessarily a Taylor traitor at x=y. It is evident, in fact, that regardless of c, the formal Taylor series for g(x),

(4) $$\sum_{k=0}^{\infty} \frac{g^{(k)}(y)}{k!}(x-y)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)+cT^{(k)}(0)}{k!}(x-y)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!}(x-y)^k = f(x),$$

converges to f(x) rather than to g(x).

In this way, given any function f(x) which is Taylor friendly at a point y, one can construct a whole set *S* of distinct functions, each of which, at x = y, possesses all derivatives and gives rise to a convergent Taylor series. However, all such series are identical and converge to a common function which differs from any member of *S*! Notice that **the derivatives of g(x) at x = y coincide with those of f(x) and thus no longer need to be null**. Moreover, by proper choice of c, **the discrepancy between g(x) and its formal Taylor series can be made as large as desired**. It therefore starts looking like 'normal' functions with 'well-behaved' Taylor expansions might be a minority!

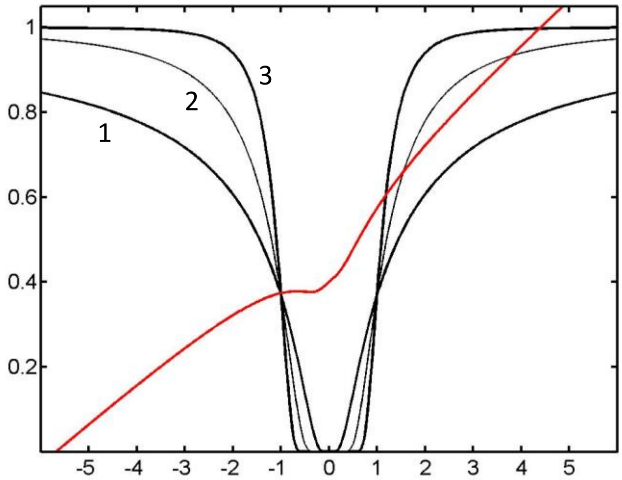

Fig.1. Examples of Taylor traitor functions
Black: $F_k(x)$ of Eq.(5) for k = 1,2,3.
Red: A combination $g(x) = (x+4)/10+F_1(x)/5$.

To make the situation even worse, notice that all the functions

(5) $$F_k(x) = \exp(-|x|^{-k}),\ k = 1,2,3,\dots,$$

[1] Note added in April 2019: This particular case is now discusssed also in the Wikipedia article [1]. It was not so when this article was first published, in January 2006.

exhibit at x = 0 qualitatively the same behavior: all their derivatives exist and are equal to zero (Appendix B). Consequently, any function of the form

(6) $\sum_{k=1}^{n} c_k F_k(s_k x)$

with constant coefficients c_k and s_k behaves at x = 0 like the F(x) of our first example and has there a vanishing formal Taylor series. The proof is in Appendix B.

This extends even to convergent infinite series (n→∞) having Taylor traitors as their terms, giving rise to a set of Taylor traitors of a staggering complexity. Since, except for fortuitous cancellations, the sum of a Taylor friendly function with *any* Taylor traitor is again a Taylor traitor, the traitors prevail by an amazingly wide margin.

An application to physics

The Author had to face nasty Taylor traitors for the first time when studying thermodynamic functions by the methods of statistical physics. In that context the temperature dependence of physical quantities is often expressed through combinations of functions of the Boltzmann type [5]:

(7) $\exp(-E/kT) \equiv F_1(x)$, with $x = kT/E$

with T denoting the absolute temperature, k the Boltzmann constant and E the energy differences between the system's energy levels. This regards in particular the partition functions [6] of statistical physics and virtually all thermodynamic functions which can be derived from them.

As one approaches absolute zero temperature, temperature derivatives of any order of any such physical quantity approach zero. The fact that functions (7) are Taylor traitors at T = 0 implies that it would be rather hard to estimate/predict the behavior of physical systems at low or moderate temperatures by extrapolating ultra-low temperature data since all simple polynomial extrapolations [7] are bound to fail.

In general, the existence of Taylor traitors means that **no simple polynomial extrapolation of empirical data is fail-safe** (this includes cosmological extrapolations to a Big Bang [8]). Not even if ALL the derivatives were PERFECTLY known, and not just rough estimates for the very first few! In particular, any extrapolation to something like the origin of our time is probably a nonsense.

Appendix A: Colloquial review of the trappings of Taylor series expansions

The basic idea behind the Taylor series is extremely simple and has to do with approximation theory [9].

If, having selected a fixed point x = y within the interval **I** = (a,b), we replace f(x) by f(y), we have a zero's order approximation which, given the continuity of f(x) in **I**, can not be quite bad in at least some small neighborhood of y.

Since f(x) has in **I** a derivative, we can go a step further and incorporate the slope of f(x) at y into our approximation, replacing f(x) by the expression f(y)+f'(y)(x-y). For 'reasonable' functions f(x) we expect such a linear approximation to be much better than the previous one.

When f(x) has in **I** derivatives up to the n-th order, one can proceed along this way and find an n-th order polynomial p(x) whose first n+1 derivatives (including the 0-th) coincide at y with those of f(x). It turns out that such a polynomial is unique and that it is given by the first term on the right-hand-side of Eq.(2).

Again, for 'reasonable' functions f(x) one expects that as n increases, the polynomials p(x) become an ever better approximation to f(x). When f(x) is expressed as the sum of p(x) and a remainder R(x) like in Eq.(2), this expectation amounts to saying that when n increases to infinity, R(x) converges to zero.

Note: Both p(x) and the remainder R(x) depend also upon n and y which should appear as their indices. Considering that this is evident from the context, we drop these indices in order to enhance readability.

Alas, **our intuition is often fallible and therefore cannot be accepted as a mathematical proof**.
A rigorous analysis of the problem shows that our intuition can in this case crash in two ways:

(1) There may be a convergence of R(x) to zero for a range of x-values close enough to y, but when the distance |x-y| exceeds a *radius of convergence*, R(x) typically decreases until some value of n and then starts to diverge to infinity!

Nice and well-understood examples of this kind of behavior can be found among functions which have an analytical continuation in the complex plane with a discrete set of poles. In such cases the radius of convergence equals the distance between the *complex* point (y,0) and the nearest complex pole. For example, the function $(1+x^2)^{-1}$ admits an analytical continuation with poles at +i and -i, so that its Taylor expansion around, let us say, y=1 converges only for x lying in the interval $(1-\sqrt{2}, 1+\sqrt{2})$.

In any case, when R(x) does not converge to zero, the limit function u(x) of Eq.(1) does not exist and there is nothing more to discuss.

(2) A more tricky case arises when all the derivatives of f(x) at y exist but are all zero and, consequently, the "approximating" function u(x) of Eq.(1) also exists but is identically null.

A naive student might conclude that, in this case, f(x) must be zero, too. The functions of Eq.(5) illustrate that this is not at all true: In fact, they are equal to zero only at the origin and nowhere else, and yet all their derivatives at the origin are null!

For pedagogical purposes and for the completeness of this exposition, let us now explicitly state the Taylor-series theorem using the two most common forms for the remainder:

Let f(x) possess continuous derivatives up to order n in the whole interval [y,x]. Then f(x) can be written as in Eq.(2) and there exist in [y,x] values θ and η such that

(8a) Lagrange form [3] $$R_{n,y}(x) = \frac{(x-y)^n}{n!} f^{(n)}(\theta)$$

(8b) Cauchy form [4] $$R_{n,y}(x) = \frac{(x-y)(x-\eta)^{n-1}}{(n-1)!} f^{(n)}(\eta)$$

Appendix B: Derivatives of $F_k(x)$ at the origin

First, let us prove that, for any polynomial P(z) and any k = 1, 2, 3, ...

(B1) $$\lim_{z\to\infty} P(z) \exp\left(-z^k\right) = 0$$

Let n be the degree of P(z) and assume z > 0. From the power-series expansion of exp(z) it follows that

(B2) $$\exp(z^k) > \sum_{m=0}^{n+1} \frac{z^{mk}}{i!} = Q(z) > 0 \quad \Rightarrow \quad \exp(-z^k) < 1/Q(z),$$

where Q(z) is the approximating polynomial of degree k(n+1). Since the degree of Q(z) is larger than that of P(z), we have

(B1) $\lim_{z\to\infty} P(z)/Q(z) = 0$

Combining this with (B2), one obtains

(B4) $\lim_{z\to\infty} |P(z)\exp(-z^k)| < \lim_{z\to\infty} |P(z)|/Q(z) \Rightarrow \lim_{z\to\infty} |P(z)\exp(-z^k)| = 0,$

which implies (B1).
Consider now the derivatives of the functions defined by Eq.(5), assuming x > 0.
We claim that they are all of the type

(B5) $\frac{d^\nu}{dx^\nu} F_k(x) = \exp\left(-\frac{1}{x^k}\right) P_\nu\left(\frac{1}{x}\right),$

where $P_\nu(z)$ is a polynomial.

The claim is certainly true for $\nu = 0$ since then P(z) = 1, regardless of k. Assume therefore that it is true for some ν and proceed to prove it for ν+1:

(B6) $\frac{d^{\nu+1}}{dx^{\nu+1}} F_k(x) = \frac{d}{dx}\left\{\exp\left(-\frac{1}{x^k}\right) P_\nu\left(\frac{1}{x}\right)\right\} = \exp\left(-\frac{1}{x^k}\right)\left\{\frac{k}{x^{k+1}} P_\nu\left(\frac{1}{x}\right) - \frac{1}{x^2} P'_\nu\left(\frac{1}{x}\right)\right\},$

where $P'_\nu(z)$ is the derivative of $P_\nu(z)$ which, of course, is also a polynomial in z. Substituting 1/x by z, one verifies that the above claim holds and obtains a recursive relation for the polynomials:

(B7) $P_{\nu+1}(z) = kz^{k+1} P_\nu(z) - z^2 P'_\nu(z).$

Returning to (B5), using again the substitution z = 1/x, and applying (B1), one obtains

(B8) $\lim_{z\to\infty+} \frac{d^\nu}{dx^\nu} F_k(x) = \lim_{z\to\infty+} \exp(-z^k) P_\nu(z) = 0.$

Since the functions defined by Eq.(5) are all even, their derivatives at -x are

(B9) $\frac{d^\nu}{dx^\nu} F_k(-x) = -\frac{d^\nu}{dx^\nu} F_k(x)$

This makes it possible to extend (B8) to the limit from the right

(B10) $\lim_{z\to\infty-} \frac{d^\nu}{dx^\nu} F_k(x) = -\lim_{z\to\infty-} \frac{d^\nu}{dx^\nu} F_k(-x) = -\lim_{z\to\infty+} \frac{d^\nu}{dx^\nu} F_k(x) = 0.$

Since the limits from the right and from the left are both null, we have finally the desired result

(B11) $\lim_{z\to\infty} \frac{d^\nu}{dx^\nu} F_k(x) = 0.$

It is interesting to notice that all the functions $F_k(x)$ are continuous and have **continuous derivatives of any order** at x = 0. Despite the presence of the absolute value in their definition, the functions therefore do not exhibit any type of singularity at the origin, except for the fact that they are Taylor traitors there. As such, they can be used also as a nice exercise on singularities.

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History of this document

This online document was updated on 25 Apr 2019 by converting it from html format to pdf, correcting a few typos and also correcting a minor (inconsequential) error in equations (B6) and (B7). .