An Abel's identity and its Corollaries

Stanislav Sykora, Extra Byte, www.ebyte.it
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This educational article shows the versatility of an 1826 identity due to Niels Hendrik Abel by using it to derive a large number of corollaries, not all of which are particularly well-known. The author thinks that a brief review of what one can do with Abel's identity might be useful.

Keywords: math, identity, Abel identity, proof, recursion, binomial identity, binomial decomposition

Introduction

The identity (AI) proved by Niels Hendrik Abel [1, 2], can be written as

$$(x+a)^n = x^n + a\sum_{k=1}^{n} C(n,k)(a-kb)^{k-1}(x+kb)^{n-k}, \qquad (1)$$

where C(n,k) stands for the binomial coefficient and, of course, n and k are integers.
It holds for any values of x, a, and b belonging to a commutative ring with unity.

Setting $b = 0$ in (1) produces the binomial identity (henceforth BI), which explains Abel's interest in AI as a generalization of the BI. Unfortunately, in later expositions [3, 4], AI is usually presented with the fixed choice $b = -1$, thus severing the link between the two identities. In (1) the symbols α, β appearing in the original text were replaced with a, b, respectively, and the current symbols for summation and for combinatorial coefficients are used. Otherwise, (1) respects exactly Abel's formulation.

The fact that only the settings $b = \pm 1$ have found popularity is probably due to the fact that (1) is invariant under the simultaneous scaling of x, a, c which sets x' = cx, a' = ca, b' = cb. This allows to normalize one of the three values and thus remove a trivial invariance. However, it obscures the case of $b = 0$ (while keeping x and a nonzero), a fact which was perhaps perceived as being of minor importance, since the BI to which it leads can be easily proved independently.

One can transform (1) in an amazing number of ways. Of these, perhaps the only one that looks trivial arises from the setting $a = 0$. One must also keep in mind that the three variables need not be integer, as long as they belong to some commutative ring. They can be, therefore, endowed with continuity (such as real or complex numbers). Consequently one may apply to any of them limit expressions, and/or apply derivatives to both sides of the equation, and thus spawn still more identities. Another fact one should bear in mind is that there is no objection to their being functions of n.

The author's interest in this matter started some time ago, after having 'discovered' an unusual looking decomposition of n^n while studying endomorphisms on finite sets[1]. It seemed to be impossible to find in the literature and all the help I could get from those who might know consisted in the advice 'use A=B'. Now, A=B is a marvelous book about systematic and even automated ways of proving hypergeometric series and combinatorial identities. It is absolutely worth reading and using and indeed it was up to the task of proving my identity. The problem, for me, was that I already had a proof, though a bit complicated one. I just wanted to know more about where the identity belonged. Whether it had a family, so to say.

In that respect I have remained unsatisfied, until I found out Abel's identity, initially in [4] and later also the original article [2], and realized that 'my' identity had an infinity of siblings, all children of AI.

[1] There are n^n possible mappings of a set of n elements into itself (each element can map into any one, including itself. This is what confers the numbers n^n a special mathematical meaning.

Special cases obtained through direct substitutions

We have already mentioned the settings b =0, which leads to BI. In what follows, we will not substitute b by any special values; let us just keep in mind that the most interesting special values for this ariable are b = ±1 and, sometimes, also b = ±n or b = ±1/n, and b = ±I, the complex unity.

Since a = 0 generates a triviality, let us try a = 1, which gives

$$(x+1)^n - x^n = \sum_{k=1}^{n} C(n,k)(1-kb)^{k-1}(x+kb)^{n-k}, \qquad (2)$$

leading to interesting decompositions for 2^n-1 (x = 1), 3^n-2^n (x = 2), etc.

Setting x = 0 in (1), one obtains

$$a^n = a\sum_{k=1}^{n} C(n,k)(a-kb)^{k-1}(kb)^{n-k}, \qquad (3)$$

Setting further a = 1 leads to decompositions of unity: for[2] n > 0,

$$1 = \sum_{k=1}^{n} C(n,k)(1-kb)^{k-1}(kb)^{n-k}. \qquad (4)$$

Setting a = n, instead, one obtains decompositions of n^n:

$$n^n = \sum_{k=1}^{n-1} C(n,k)n\,(n-kb)^{k-1}(bk)^{n-k}. \qquad (5)$$

Other types of substitutions can bind together the value of x with that of a.
For example, setting x = a or x = -a/2 in (1) leads to, respectively,

$$(2^n-1)a^n = \sum_{k=1}^{n} C(n,k)a(a-kb)^{k-1}(a+kb)^{n-k}, \qquad (6)$$

$$[1-(-1)^n](a/2)^n = \sum_{k=1}^{n} C(n,k)a(a-kb)^{k-1}(kb-a/2)^{n-k}, \qquad (7)$$

with the rather interesting special setting of a = 2 in the last case.

Notice also the fact that whenever the l.h.s. is independent of b, the right hand sides obtained with different settings of b must be identical. Thus, in general, (1) implies that for a ≠ 0 the identity

$$\sum_{k=1}^{n} C(n,k)(a-kb)^{k-1}(x+kb)^{n-k} = \sum_{k=1}^{n} C(n,k)(a-kc)^{k-1}(x+kc)^{n-k} \qquad (8)$$

holds for any values of x, a, b, and c.

Identities obtained through limits

For a ≠ 0, one can rewrite (1) as

$$((x+a)^n - x^n)/a = \sum_{k=1}^{n} C(n,k)(a-kb)^{k-1}(x+kb)^{n-k}. \qquad (9)$$

As already explained, we are entitled to take the limit of the latter identity for a → 0.
When done, the result, valid for any x and b, is

$$\boxed{nx^{n-1} = \sum_{k=1}^{n} C(n,k)(-kb)^{k-1}(x+kb)^{n-k} \qquad (10)}$$

which, incidentally, proves that (8) is valid also for a = 0.

The most obvious special case of (10) results from setting x = 0. When n > 1, one obtains

$$0 = \sum_{k=1}^{n} C(n,k)(-1)^{k-1}(kb)^{n-1}. \qquad (11)$$

Another interesting special case arises from (10) by replacing b with bx.
After a simple manipulation, this leads to decompositions of n:

$$n = \sum_{k=1}^{n} C(n,k)(-kb)^{k-1}(1+kb)^{n-k}. \qquad (12)$$

[2] The cases of n=0 in many of the identities in this Note are special and require a conventional setting.

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For x = n, instead, the result is a generic decomposition[3] of n^n:

$$n^n = \sum_{k=1}^{n} C(n,k)(-kb)^{k-1}(n+kb)^{n-k}. \qquad (13)$$

It is interesting to compare (13), for x = n, with the BI formula for $((n+b)-b)^n$:

$$n^n = \sum_{k=0}^{n} C(n,k)(-b)^{k}(n+b)^{n-k}.$$

A limit of a different kind is obtained from (2) by setting x = 1/n and letting n go to infinity:

$$\lim_{n\to\infty} \sum_{k=1}^{n} C(n,k)(1-kb)^{k-1}(kb+(1/n))^{n-k} = e. \qquad (14)$$

The interesting thing about this expression is that the absolute values of the individual terms in the summation grow very rapidly when n becomes large, while their sum converges to the Euler number.

Identities obtained through derivatives

Taking the derivative of both sides with respect to 'x' leads to a restatement of (1) with n substituted by (n-1) and thus offers nothing new.

Assuming a ≠ b, the derivative with respect to 'a' leads to:

$$\boxed{n[(x+a)^{n-1}-(x+b)^{n-1}]/(a-b) = \sum_{k=2}^{n} C(n,k)k(a-kb)^{k-2}(x+kb)^{n-k}. \qquad (15)}$$

Notice that, unlike in the previous cases, the l.h.s. of (15) now depends also on b.

Since the l.h.s. of (15) does not change when the variables (a, b) are exchanged, neither can the r.h.s. Hence, for any x, a, b

$$\sum_{k=2}^{n} C(n,k)k(a-kb)^{k-2}(x+kb)^{n-k} = \sum_{k=2}^{n} C(n,k)k(b-ka)^{k-2}(x+ka)^{n-k}. \qquad (16)$$

Setting x = 0 in (15), and assuming a ≠ b, one obtains

$$n(a^{n-1}-b^{n-1})/(a-b) = \sum_{k=2}^{n} C(n,k)k(a-kb)^{k-2}(kb)^{n-k}. \qquad (17)$$

An elementary expansion of the l.h.s. of this equation gives

$$b^{n-2}\sum_{m=0}^{n-2} (a/b)^m = (1/n)\sum_{k=2}^{n} C(n,k)k(a-kb)^{k-2}(kb)^{n-k}, \qquad (18)$$

which, thanks to another limit transition, holds also for a/b = 1. Setting b = 1, this allows us to rewrite a geometric series in terms of a binomial one

$$\sum_{k=0}^{n-2} a^k = (1/n)\sum_{k=2}^{n} C(n,k)k(a-k)^{k-2}(k)^{n-k}. \qquad (19)$$

In particular, (19) tells us that, when |a| < 1, the following limit exists[4]:

$$\lim_{n\to\infty}(1/n)\sum_{k=2}^{n} C(n,k)k(a-k)^{k-2}(k)^{n-k} = 1/(1-a). \qquad (20)$$

Another interesting setting in (15) is x = -a. For n>1, it gives

$$n(b-a)^{n-2} = \sum_{k=2}^{n} C(n,k)k(a-kb)^{k-2}(kb-a)^{n-k}. \qquad (21)$$

Finally, the derivative with respect to b leads to this rather cumbersome identity:

$$n(n-1)[(x+b)^{n-2}-(a-nb)^{n-2}] =$$
$$= \sum_{k=2}^{n-1} C(n,k)k(a-kb)^{k-2}(x+kb)^{n-1-k}[(k-1)(x+kb)-(n-k)(a-kb)]. \qquad (22)$$

It verifies numerically, but at present it is not clear which of its special cases might be of interest.

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[3] It was this identity, with b=-1, which originally caught the author's attention.

[4] The same comment as the one following eq.(14) is applicable also here.

Conclusions

Abel's identity in its original form is a formidable source of many non-trivial identities. Special versions of these could be probably proved relatively easily using modern automated computer-proof methods [5] but that might hide the broader perspective comprising the whole family.

The fact that Abel's identity (AI) contains three 'arbitrary' variables which need only to be members of a commutative ring with unity (and thus, for example, real or complex) makes it very versatile and makes it possible to derive further identities by means of direct substitutions as well as limit transitions and derivatives. In this it resembles the binomial identity (BI) which, in fact, is one of its special cases.

Given the number of distinct special identities following from AI, it is not easy to wrap them up in a mnemonically easy way. Those investigated here can be all cast in the form of a weighed sum over the binomial coefficients (a binomial decomposition):

$$A(n,x,a,b) = \sum_{k=0}^{n} C(n,k)w(n,k,x,a,b). \qquad (22)$$

Here w(n, k, x, a, b) are the decomposition weights and A(n, x, a, b) is the sum (A standing for Abel). We have included in (22) also the term with k = 0 because it appears in BI and is therefore better compatible with a more general scheme. To avoid any problems one can simply adopt the convention that w is zero whenever the index is outside of the required range.

In many cases the argument 'b' appears on the right hand side of (22) but not on the left hand side. Such cases implicate a general identity of the following type, valid for any x, a, b, and c:

$$\sum_{k=0}^{n} C(n,k)w(n,k,x,a,b) = \sum_{k=0}^{n} C(n,k)w(n,k,x,a,c). \qquad (23)$$

Table I: Some interesting identities derived from Abel's

Notes: x, a, b are any elements of a commutative ring with unity (e.g., integer, rational, real, or complex numbers). Common settings for b are +1 and -1. Yellow background marks the principle generic cases. The sequences submitted to OEIS are just representative examples of the infinity (literally) of those possible.

w(n, k, x, a, b)	k =	A(n, x, a) = Σ_k w(n, k, x, a, b)	Eq.	Note	OEIS refs
$a(a-kb)^{k-1}(x+kb)^{n-k}$	1..n	$(x+a)^n - x^n$	(1)	original AI	
$(1-kb)^{k-1}(x+kb)^{n-k}$	1..n	$(x+1)^n - x^n$	(2)	a = 1	
$a(a-kb)^{k-1}(kb)^{n-k}$	1..n	a^n	(3)	x = 0	
$(1-kb)^{k-1}(kb)^{n-k}$	1..n	1	(4)	x = 0, a = 1	[8, 9]
$n(n-kb)^{k-1}(kb)^{n-k}$	1..n	n^n	(5)	x = 0, a = n	[10,11]
$a(a-kb)^{k-1}(a+kb)^{n-k}$	1..n	$(2^n-1)a^n$	(6)	x = a	[12,13]
$a(a-kb)^{k-1}(kb-a/2)^{n-k}$	1..n	$[1-(-1)^n](a/2)^n$	(7)	x = –a/2	
$(-kb)^{k-1}(x+kb)^{n-k}$	1..n	$n.x^{n-1}$	(10)	$\lim_{a\to 0}$ AI	
$(-1)^{k-1}(kb)^{n-1}$	1..n	1 when n=1, 0 otherwise	(11)	x = 0	[14]
$(-kb)^{k-1}(1+kb)^{n-k}$	1..n	n	(12)	x = 1	[15,16]
$(-kb)^{k-1}(n+kb)^{n-k}$	1..n	n^n	(13)	x = n	[17,18]
$k(a-kb)^{k-2}(x+kb)^{n-k}$	2..n	$n[(x+a)^{n-1}-(x-b)^{n-1}]/(a-b)$	(15)	∂AI/∂a	
$k(a-kb)^{k-2}(kb)^{n-k}$	2..n	$n(a^{n-1}-b^{n-1})/(a-b)$	(17)	x = 0	
$k(a-k)^{k-2}(k)^{n-k}$	2..n	$n\,\Sigma_{m=0,n-2}\, a^m$	(19)	x = 0, b = 1	[19]
$k(a-kb)^{k-2}(kb-a)^{n-k}$	2..n	$n(b-a)^{n-2}$	(21)	n>1, x =-a	[20,21]

Appendix A: the original proof of (1) by Niels Hendrik Abel

This page, and the following one, were scanned from the original article.

Beweis eines Ausdruckes, von welchem die Binomial-Formel ein einzelner Fall ist.

(Von Herrn *N. H. Abel.*)

Der Ausdruck ist folgender:

$$(x+\alpha)^n = x^n + \frac{n}{1}\alpha.\,(x+\beta)^{n-1} + \frac{n.n-1}{1.\;2}\alpha(\alpha-2\beta)(x+2\beta)^{n-2}\dots\dots$$
$$+\frac{n.n-1\dots\dots(n-\mu+1)}{1.\;2\dots\dots\mu}\alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n-\mu}$$
$$+\frac{n}{1}\alpha\big(\alpha-(n-1)\beta\big)^{n-2}\big(x+(n-1)\beta\big)+\alpha(\alpha-n\beta)^{n-1}.$$

x, α und β sind beliebige Gröfsen, n ist eine ganze positive Zahl.

Wenn $n=0$: so giebt der Ausdruck

$$(x+\alpha)^0 = x^0;$$

wie gehörig, Nun kann man, wie folgt, beweisen, dafs der Ausdruck, wenn er für $n=m$ Statt findet, auch für $n=m+1$, also allgemein, gilt.

Es sei

$$(x+\alpha)^m = x^m + \frac{m}{1}\alpha(x+\beta)^{m-1} + \frac{m.m-1}{1.\;2}\alpha(\alpha-2\beta)(x+2\beta)^{m-2}\dots\dots$$
$$+\frac{m}{1}\alpha\big(\alpha-(m+1)\beta\big)^{m-2}\big(x+(m-1)\beta\big)+\alpha(\alpha-m\beta)^{m-1}.$$

Man multiplicire mit $(m+1)dx$ und integrire, so findet man:

$$(x+\alpha)^{m+1} = x^{m+1} + \frac{m+1}{1}\alpha(x+\beta)^m + \frac{(m+1)m}{1.\;2}\alpha(\alpha-2\beta)(x+2\beta)^{m-1}\dots\dots$$
$$\dots\dots+\frac{m+1}{2}\alpha(\alpha-m\beta)^{m-1}(x+m\beta)+C,$$

wo C die willkürliche Constante ist. Um ihren Werth zu finden, sei

$$x=(m+1)\beta,$$

so geben die beiden letzten Gleichungen:

$$\big(\alpha-(m+1)\beta\big)^m = (-1)^m\Big[(m+1)^m\beta^m - m^m\alpha\beta^{m-1} + \frac{m}{2}(m-1)^{m-1}\alpha(\alpha-2\beta)\beta^{m-1}$$
$$-\frac{m.m-1}{2.\;3}\alpha(\alpha-3\beta)^2(m-2)^{m-2}\beta^{m-3}\dots\dots\Big],$$

$$\big(\alpha-(m+1)\beta\big)^{m+1} = (-1)^{m+1}\Big[(m+1)^{m+1}\beta^{m+1} - (m+1)m^m\alpha\beta^m$$
$$+\frac{(m+1)m}{2}(m-1)^{m-1}\alpha(\alpha-2\beta)\beta^{m-1}\dots\dots\Big]+C.$$

Multiplicirt man die erste Gleichung mit $(m+1)\beta$, und thut das Product zur zweiten, so findet man:

$$C=\big(\alpha-(m+1)\beta\big)^{m+1}+(m+1)\beta\big(\alpha-(m+1)\beta\big)^m, \text{ oder}$$
$$C=\alpha\big(\alpha-(m+1)\beta\big)^m.$$

Daraus folgt, dafs der zu beweisende Ausdruck auch für $n=m+1$ Statt findet. Er gilt aber für $n=0$; also gilt er auch für $n=0, 1, 2, 3$ etc., das heifst: für jeden beliebigen ganzzahligen und positiven Werth von n.

Setzt man $\beta=0$, so bekommt man die Binominal-Formel.

Setzt man $\alpha=-x$, so findet man:

$$0=x^n-\frac{n}{1}x(x+\beta)^{n-1}+\frac{n.n-1}{2}x(x+2\beta)^{n-1}$$
$$-\frac{n.(n-1)(n-2)}{2.\;3}x(x+3\beta)^{n-1}\dots\dots,$$

oder, wenn man mit x dividirt,

$$0=x^{n-1}-\frac{n}{1}(x+\beta)^{n-1}+\frac{n.n-1}{2}(x+2\beta)^{n-1}$$
$$-\frac{n.(n-1)(n-2)}{2.\;3}(x+3\beta)^{n-1}\dots\dots,$$

wie auch sonst schon bekannt ist; denn das zweite Glied dieser Gleichung ist nichts anderes, als

$$(-1)^{n-1}\Delta^n\left(x^{n-1}\right),$$

wenn man die constante Differenz gleich β setzt.

Note: this proof, dated 1826, looks very 'modern' since it establishes the identity of the two expressions by (a) proving that they both satisfy the same recursion[5] in terms of n, and (b) that the starting values of the two recursions are also the same. Today, this could be viewed as an example of *'Sister Celine's method'* though, in this author's opinion, that would be an oversimplification, somewhat unfair both towards the mathematicians of Abel's period who used the 'method' routinely, and towards Sister Celine (Mary Celine Fasenmyer [6]) who contributed primarily to the theory of hypergeometric functions.

[5] Tho establish the recurrence, Abel uses an analytical approach (integration), typical of the times.

Appendix B: synopsis of the proof of Abel's identity by Ekhad and Majewicz

In reference [4], identity (1) is proved for the special form with b = -1, by showing the equality between

$a_n(r,s) = \sum_{k=1}^{n} F_{n,k}(r,s)$, where $F_{n,k}(r,s) = C(n,k)(r+k)^{k-1}(s-k)^{n-k}$, and

$b_n(r,s) = (s+r)^n/r$,

with r, s being two elements of an Abelian ring with unity (note that here $a_n(r,s)$ and $b_n(r,s)$ have nothing in common with the a and b of (1)). The proof proceeds as follows.

1) Let $G_{n,k}(r,s) = (s-n)\ C(n-1,k-1)\ (r+k)^{k-1}(s-k)^{n-k-1}$.
2) Check explicitly that the following statement is true (this is laborious, but straightforward): $F_{n,k}(r,s) - s\ F_{n-1,k}(r,s) - (n+r)\ F_{n-1,k}(r+1,s-1) + (n-1)(r+s)\ F_{n-2,k}(r+1,s-1) = G_{n,k}(r,s) - G_{n,k+1}(r,s)$.
3) Sum both sides of the last equation from k = 0 to n, exploiting the telescoping on the r.h.s. The result is the following second-order recurrence for $a_n(r,s)$: $a_n(r,s) - s\ a_{n-1}(r,s) - (n+r)\ a_{n-1}(r+1,s-1) + (n-1)(r+s)\ a_{n-2}(r+1,s-1) = 0$.
4) Check explicitly that the same recurrence holds also for $b_n(r,s)$.
5) Check that the starting terms of the two recurrences coincide (1/r for n=0, (r+s)/r for n = 1).
6) Consequently, the two sequences are identical, QED.

This author's notes: Roughly speaking, this proof is based on induction and recurrence, much as the original, but it starts from a less general formula. The telescoping 'trick' in step (3) is very nice, but the statement in (2) which leads to it looks like a disconnected piece of truth – a price paid for avoiding analytical tools.

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[6] Wikipedia, Mary Celine Fasenmyer (Sister Celine).
[7] The On-Line Encyclopedia of Integer Sequences, OEIS, published electronically at http://oeis.org.

References [8-23] are sequences which were registered in OEIS. Please, follow the links to see each sequence, its values up to n = 100, and its graphs.

[8] Eq.(4). A244116 (coefficients w(n,k)), A244117 (terms C(n,k)*w(n,k)); b = 1.
[9] Eq.(4). A244118 (coefficients w(n,k)), A244119 (terms C(n,k)*w(n,k)); b = -1.
[10] Eq.(5). A244120 (coefficients w(n,k)), A244121 (terms C(n,k)*w(n,k)); b = 1.
[11] Eq.(5). A244122 (coefficients w(n,k)), A244123 (terms C(n,k)*w(n,k)); b = -1.
[12] Eq.(6). A244124 (coefficients w(n,k)), A244125 (terms C(n,k)*w(n,k)); b = 1.
[13] Eq.(6). A244126 (coefficients w(n,k)), A244127 (terms C(n,k)*w(n,k)); b = -1.
[14] Eq.(11). A244128 (coefficients w(n,k)), A244129 (terms C(n,k)*w(n,k)); b = 1.
[15] Eq.(12). A244130 (coefficients w(n,k)), A244131 (terms C(n,k)*w(n,k)); b = 1.
[16] Eq.(12). A244132 (coefficients w(n,k)), A244133 (terms C(n,k)*w(n,k)); b = -1.
[17] Eq.(13). A244134 (coefficients w(n,k)), A244135 (terms C(n,k)*w(n,k)); b = 1.
[18] Eq.(13). A244136 (coefficients w(n,k)), A244137 (terms C(n,k)*w(n,k)); b = -1.
[19] Eq.(19). A244138 (coefficients w(n,k)), A244139 (terms C(n,k)*w(n,k)); a = b = 1.
[20] Eq.(21). A244140 (coefficients w(n,k)), A244141 (terms C(n,k)*w(n,k)); a=2, b = 1.
[21] Eq.(22). A244142 (coefficients w(n,k)), A244143 (terms C(n,k)*w(n,k)); a=1, b = 2.

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