Blazys Expansions and Continued Fractions

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A few years ago Don Blazys caused a ripple of excitation among numerologists by presenting a real number which, applying an iterative computation recipe, produced all prime numbers. The procedure is one of an infinity of possible mappings between subsets of real numbers and subsets of integer sequences. In this particular case, the source set is that of *irrational real numbers greater than 1*, and its image is the set of *sequences of non-decreasing natural numbers*. It turns out that the mapping, denoted here as bx(x), is a *bijection* between the two sets, thus enabling the existence of an *inverse mapping* bf(s) which, in addition, can be cast as a *special type of generalized continued fractions*. This article presents the definitions, the proofs of the bijection and the pertinent algorithms. It also analyses some simple properties of these mappings.

Keywords: math, sequence, mapping, irrational, number, continued fraction, convergence, bijection

I. Introduction

Some time ago, Don Blazys presented [1] a real number, which - applying a specific algorithm - generates the complete sequence of prime numbers. The value of this **Blazys' constant,** was

(1) B = 2.566 543 832 171 388 844 467 529 … (OEIS A233588)

Following a Blazys' recipe (see below) this constant indeed generates correctly the first 16 prime numbers (from 2 to 53). The original article played on numerological mysticism by claiming that the number matched a root of the following ad-hoc equation:

(2) $(\sin(\sqrt{x})^{-1}-1)^{-1} - \ln(\ln((3x+2)/(x+1)))^{-1} = \pi^2$.

The latter claim, however, turns out to be in error, because the root of the equation evaluates to

(3) 2.566 543 832 172 425 504 475 092 …,

and therefore fails to generate primes after p = 29 (next entry being 39), a fact initially attributed to insufficient numeric precision (the relative error is indeed amazingly small).

It seems that Don Blazys[1] tired of searching for a better match and stopped promoting the matter. The whole topic also slowly drifted out of focus in numerological circles.

However, the procedure (henceforth called **Blazys' expansion**) is by itself a well-defined mapping of a class of real numbers into a class of integer sequences and, once its domain and image sets are properly delimited, it becomes a useful mapping with educational and, hopefully, mathematical potential. It turns out that it has an inverse which coincides with a special type of generalized continued fractions [2,3]. Since it consists in the iteration of very simple mappings, it also bears some conceptual similarity to infinite tetration towers [4] and to binary iterated powers [5]. These are good enough reasons to study it in more detail.

[1] Don Blazys is an amateur mathematician, known for this primes-generating constant and, even more so, for his somewhat controversial proof of Beal's conjecture. His website is http://donblazys.com/ - and most of the discussions surrounding him - can be found on the web. Despite the title of this article, I am NOT taking any position at all on Don Blazys as a person, and I flatly refuse to get involved in any discussion of his writings. Mathematics is not about opinions and I did not study his writings beyond what appears here. I am at a loss why he did not attempt to work out the math side of his constant, a side that I find interesting. But, since it was his article that brought my attention to the matter, I have no problem associating his name with the related mappings.

II. Definition of Blazys' expansion

The Blazys' algorithm for converting a positive real number x into a sequence of natural numbers **s** = $\{s_1, s_2, s_3, \ldots\}$ can be summed-up as:

1. Set k=1 and $x^{(k)} = x$.
2. Set s_k to the integer part of $x^{(k)}$, i.e., $s_k = \text{floor}(x^{(k)})$.
3. Set $x^{(k+1)} = 1/(x^{(k)}/s_k - 1) = s_k/(x^{(k)} - s_k) = s_k/(x^{(k)} - \text{floor}(x^{(k)}))$.
4. Increment k by 1 and iterate back to (2), indefinitely[2].

Applied to the Blazys' constant (1) it generates the prime numbers sequence:

s = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, ...} OEIS A000040.

In general, the only problem with this algorithm might arise in step (3), where it might fail due to division by zero, should $x^{(k)}$ become an exact integer. It is easy to show that this will happen if, and only if, r is either smaller than 1 (in which case c becomes immediately 0) or when r is rational.

For irrational real numbers greater than 1, the procedure is always applicable, even though in practice one might sometimes encounter numerical precision concerns. Since $0 < x^{(k)} - \text{floor}(x^{(k)}) < 1$, it follows that $s_{k+1} \geq s_k$, i.e., that the resulting sequence terms are always non-decreasing.

For example, for rational numbers one might stop short of the step where the division by zero occurs and thus associate bx(x) for rational numbers with finite integer sequences. However, such a convention would be artificial and we will not pursue it here. For now, it is preferable to say simply that the expansion, henceforth called bx(x) is an injection of the set of **irrational real numbers greater than 1** into the set of **non-decreasing sequences of natural numbers**[3].

Let us see a few more Blazys' expansions; computed using the PARI freeware package [6] and my PARI scripts [7]. These symbols are used: Φ is the golden ratio [8, 9, OEIS A001622], e is the base of natural logarithms [10, 11, OEIS A001113], π is the Pi number [12, 13, OEIS A000796], and γ is the Euler number [14, 15, OEIS A001620]. We also use the underline to indicate periodic repetition of the last term.

bx(Φ)	= {1, 1, 1, 1, 1, 1, ... } ≡ {$\underline{1}$}	
bx(e)	= {2, 2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, ... }	OEIS A233583
bx(e-1)	= {1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, ... }	
bx(√e)	= {1, 1, 1, 1, 5, 9, 17, 109, 260, 2909, 3072, 3310, 3678, 6715, ... }	OEIS A233584
bx(π)	= {3, 21, 111, 113, 158, 160, 211, 216, 525, 1634, 1721, 7063, ... }	OEIS A233582
bx(π-1)	= {2, 14, 111, 113, 158, 160, 211, 216, 525, 1634, 1721, 7063, ... }	
bx(π-2)	= {1, 7, 111, 113, 158, 160, 211, 216, 525, 1634, 1721, 7063, ... }	
bx(1/γ)	= {1, 1, 2, 2, 2, 2, 4, 12, 39, 71, 83, 484, 1028, 1447, 9913, 31542, ... }	OEIS A233585
bx(2γ)	= {1, 6, 12, 19, 63, 263, 856, 2632, 7714, 9683, 888970, 1200867, ... }	OEIS A233586
bx(√2)	= {1, 2, 4, 4, 4, 4, ... } ≡ {1, 2, $\underline{4}$}	
bx(√3)	= {1, 1, 2, 2, 2, 2, ... } ≡ {1, 1, $\underline{2}$}	
bx(√5)	= {2, 8, 16, 16 16, 16, ... } ≡ {2, 8, $\underline{16}$}	
bx(√6)	= {2, 4, 8, 8, 8, 8, ... } ≡ {2, 4, $\underline{8}$}	
bx(√7)	= {2, 3, 30, 34, 111, 235, 3775, 5052, 7352, 9091, 34991, 35530, ... }	OEIS A233587
bx(√8)	= {2, 2, 4, 4, 4, 4, ... } ≡ {2, 2, $\underline{4}$}	

[2] Here, k stands for the iteration number (starting with k=1) and the superscript (k) indicates k-th iteration value of the modified number $x^{(k)}$ which, in a practical implementation, can be done in-place, using a single location.
[3] Saying 'natural numbers' underlines the fact that we intend positive integers, excluding any leading zeroes.

III. Blazys' continued fractions

The original recipe defining bx(x) can be inverted, hopefully producing a mapping of any non-decreasing sequence **s** of positive integers into an irrational real number, such that bx(bf(**s**)) = **s**.

Given a sequence **s** = $\{s_1, s_2, s_3, \ldots\}$ of the specified type, an inversion recipe can be sketched[4] as:

a) Start at some N and set k=N, $x^{(k)} = s_k$.
b) Decrement k by 1 and recalculate x according to the formula $x^{(k)} = s_k\,(1+1/x^{(k+1)})$.
c) Repeat step (b) until k=1.

The resulting value $x^{(1)}$ evidently depends on the starting value of N. The assumption is that, with increasing N, it converges to the desired value x for which bx(x) = **s** :

d) $x = \lim_{N\to\infty} x^{(1)}$.

For now, this recipe is just a strategic plan based, in step (b), on the reversal of step (3) of the Blazys' expansion algorithm, namely the inversion of the assignment $x^{(k+1)} = s_k/(x^{(k)} - s_k)$. Assuming that everything works, the result x can be formally written as a variety of a generalized continued fraction:

(4) $\mathrm{bf}(s_1, s_2, s_3, \ldots) = s_1+s_1/(s_2+s_2/(s_3+s_3/(s_4+s_4(\ldots))))$,

This is the reason why we will call the expressions bf(s) of this form **Blazys' continued fractions** for the sequence **s**. Comparing a Blazys' continued fraction bf(**s**) with the generalized continued fraction formula,

(5) $\mathrm{gf}(b_0, b_1, b_2, b_3, \ldots|\, a_1, a_2, a_3, \ldots) = b_0+a_1/(b_1+a_2/(b_2+a_3/(b_3+a_4(\ldots))))$,

we see that the former is a special case of the latter, the generalization consisting in the identification of a_k with s_k and b_k with s_{k+1}, i.e.,

(6) $\mathrm{bf}(s_1, s_2, s_3, \ldots) \equiv \mathrm{gf}(s_1, s_2, s_3, \ldots|\, s_1, s_2, s_3, \ldots)$.

Incidentally, this correspondence explains the value of bf(1, 1, 1, ...) since in this case (and only in this case) the Blazys continued fraction coincides with simple continued fraction for golden ratio [9,10].

The correspondence also answers a question regarding convergence since, according to our assumptions, a_k and b_k are nonzero integers such that $s_k = a_k \le b_k = s_{k+1}$. These are sufficient pre-requisites for a theorem [16] proving that, indeed, all Blazys' continued fractions converge to an irrational limit[5]. But we will return to this point later.

Notice also that, to guarantee the convergence, it is sufficient that the sequence **s** be monotonically non-decreasing only from some index K up. In other words, as far as convergence is concerned, there may be a finite number of leading non-zero integer terms with arbitrary values, not just non-decreasing ones.

For non-decreasing sequences, the convergence is quite fast. Even in the slowest-converging case of bf(1,1,1,1,1,...) the distance from the limit decreases by a factor of at least 0.5 in each step, but in many other cases is much faster. Consequently, evaluation to M steps guarantees even in the worst case a precision of at least M binary digits. Moreover, the standard recipe [3,17,18] for evaluating continued fractions in a single "forward" sweep is also applicable, allowing to replace points a-c) with the following ones:

A) Initialize a two-variables iteration by setting $A_{-1} = 1$, $A_0 = s_1$, $B_{-1} = 0$, $B_0 = 1$
B) Let $A_{n+1} = A_n{*}s_{n+1} + A_{n-1}{*}s_n$, and $B_{n+1} = B_n{*}\, s_{n+1} + B_{n-1}{*}\, s_n$.
C) Then the ratios $f_n = A_n/B_n$ form a convergent series such that, moreover, it is decreasing for odd n and increasing for even n, thus providing an easy test of the current precision.

[4] Later we will see a better way to do it.
[5] For completeness sake, in Section V we will anyway give an explicit proof of the convergence.

One special case which merits a separate consideration is $x_m = bf(m, m, m, \ldots) \equiv bf(\underline{m})$, a generalization of the case of golden ratio (notice that longer infinitely repeating cycles are excluded because the sequences must be non-decreasing). Clearly, x_m must satisfy the equation $x_m = m + m/x_m$ and therefore

(7) $$bf(m, m, m, \ldots) = \left(m + \sqrt{m(m+4)}\right)/2.$$

For m=1 this indeed gives the golden ratio [9,10]. For higher values of m one obtains many constants involving simple square roots of integers which are already listed in the OEIS, such as: A090388 (m=2), A090458 (m=3), A090488 (m=4), A090550 (m=5), A092294(m=6), A092290 (m=7), A090654 (m=8), A090655 (m=9), A090656 (m=10).

Let us now see a few more examples. Naturally, one can first check the reversibility of all the special cases of bx(x) listed in the preceding Section. To these, we add the following simple cases:

The original Blazys' constant corresponding to the prime numbers series, evaluated to a higher precision:

bf(2, 3, 5, 7, 11, 13, 17, 19, 23, ...) =
2.56654383217138884446752910633228575178297282870231464596973... OEIS A233588.
For comparison, see the simple continued fraction expansion for prime numbers (OEIS A064442), which evaluates to 2.313036736...

Natural numbers sequence:

bf(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...) =
1.392211191177332814376552878479816528373978385315287... = 1/(e-2) OEIS A194807.
Note: The identification of this result with 1/(e-2) can be done through a simple manipulation of the Wall's continued fraction for e [18,19].

Factorials:

bf(0!, 1!, 2!, 3!, 4!, 5!, 6!, 7!, 8!, 9!, 10!, ...) =
1.69880476767000721195269011591464043255973093664983969781741... OEIS A233589

Powers of 2:

bf(1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...) =
1.40861597973500520513236259025579520948456337368688835370392... OEIS A233590

Squares of natural numbers:

bf(1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...) =
1.22628402418269027481493710086224039619081148735362359550166... OEIS A233591

IV. Some elementary properties

From the definition it follows that

(8) $bf(s_1, s_2, s_3, \ldots) = s_1 + s_1/bf(s_2, s_3, s_4, \ldots)$ and, vice versa,
(9) $bf(s_2, s_3, s_4, \ldots) = s_1 / (bf(s_1, s_2, s_3, \ldots) - s_1)$.

Thus, for example,

bf(1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, ...) = 1+1/B, and
bf(3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 57, ...) = 1/(B-1).

Obviously, relations (8) and (9) can be iterated any number of times. We might also consider them as operators on the continued fraction value which extend/or truncate the leading portion of the corresponding sequence.

Combining equations (8) and (9) one obtains:

(10) $s_1' * bf(s_1, s_2, s_3, \ldots) = s_1 * bf(s_1', s_2, s_3, \ldots)$

and the corollaries

(11) $bf(s_1, s_2, s_3, \ldots) = s_1 * bf(1, s_2, s_3, \ldots)$, $bf(s_1, s_2, s_3, \ldots) = K * bf(Ks_1, s_2, s_3, \ldots)$.

Carrying this a step further, we have further

(12) $(s_2/s_1)* bf(s_1, s_2, s_3, s_4, \ldots) - (s_2'/s_1')* bf(s_1', s_2', s_3, s_4, \ldots) = s_2 - s_2'$

and the corollaries

(13) $s_2* bf(1, s_2, s_3, s_4, \ldots) - s_2'* bf(1, s_2', s_3, s_4, \ldots) = s_2 - s_2'$
(14) $bf(s_1, s_2, s_3, s_4, \ldots) = s_1 + (s_1/s_2) * [\, bf(1, 1, s_3, s_4, \ldots) - 1]$

These relations explain, for example, many of the features observed in the Blazys' expansions listed on page 2, such as the similarity between bx(π), bx(π-1), and bx(π-2), that between bx(√2) and bx(2√2), and the ones between bx(e), bx(e-1), bx(e-2). In the latter case, since the formula for bx(e-2) is rigorously proved via Wall's continued fraction for e [19], we can consider as proved also bx(e) and bx(e-1).

Other relations of the above kind can be derived proceeding further along this road to differences in 3, 4 etc leading terms of the sequences. The 'normalizations', for example, could be extended to any number of leading terms. Such manipulations of the sequences and the corresponding continued fractions, however, require additional considerations, due to the following phenomenon:

Consider the Blazys' expansion of π, shown on page 2. Since bx(π) = {3, 21, 111, 113, …} and therefore bf(3, 21, 111, 113, …) = π, it follows from equation 11 that bf(6, 21, 111, 113, …) = 2π. Indeed, this is fully confirmed by a numeric test and so is the inverse thereof, bx(2π) = {6, 21, 111, 113, …}. But consider the expansion of golden ratio bx(Φ) = {1, 1, 1, 1, 1, 1, … } and the continued fraction bf(1, 1, 1, 1, 1, 1, …) = Φ. Again, equation 11 guarantees that bf(2, 1, 1, 1, 1, 1, …) = 2Φ, as can be easily verified numerically. In this case, however, the inverse expansion turns out to be bx(2Φ) = {3, 12, 16, 16, 16, 16, … }, indicating that 2Φ = bf(2, 1, 1, 1, 1, 1, …) = bf(3, 12, 16, 16, 16, 16, …). We have two different continued fractions evaluating to the same number!

The reason is that the sequence {2, 1, 1, 1, 1, 1, … }, even though its Blazys' continued fraction converges, is NOT non-decreasing and thus does not belong to the image set of Blazys' expansions. Equations (8 – 14) are valid for any convergent continued fraction of the type given by equation 4, regardless of whether s(n) is monotonously non-decreasing or not, while Blazys' expansion maps an irrational number exclusively on a non-decreasing sequence.

This method of generating multiple continued fraction representations of the same number is very universal. As another example, consider bf(3, 21, 111, 113, …) = π and therefore, bf(3K, 21, 111, 113, …) = Kπ for any positive integer K (eqs.11). But the Blazys' expansion bx(Kπ) equals {3*K, 21, 111, 113, …} only for as long as K <= 7, i.e., as long as the sequence {3*K, 21, 111, 113, …} remains non-decreasing (in other words, as long as K*(x-floor(x)) < 1, with x = π). In fact, bx(8π) = {25, 188, 558, 604, 1027, …} which has no apparent similarity to bx(Kπ) for K= 1 - 7.

What all this means is that Blazys' expansion bx(x) *could be* a bijection [20,21], with bf(**s**) as its inverse, but only when the domain of bf(**s**) is restricted to *non-decreasing* sequences of natural numbers. The fact that it really is a bijection, however, is yet to be proved - so far, we have only shown, by means of numeric counter-examples, that without the restriction there could be no bijection at all.

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V. Proof of the bijection

To prove that a mapping $B:X\to S$ is a bijection [20,21], we have to show that

(i) every element x of X maps onto an element of S,
(ii) every element **s** of S has a source x in X, such that $B(x)=\mathbf{s}$,
(iii) two distinct elements of X map onto two distinct elements of S,
(iv) two distinct elements of S have two distinct sources in X.

In our case B is the Blazys' expansion bx(x), X is the set of all irrational numbers greater than 1, and S is the set of all non-decreasing sequences of natural numbers.

Point (i) is guaranteed by the *construction* of the mapping $bx(x):X\to S$ introduced in Section II.

Point (ii) is guaranteed by the *construction* of the inverse of bx(x) in Section III, and by the subsequent discussion. However, for completeness sake, let us give an explicit proof, based just on the construction recipe, rather than relying on ref.[16].

Let us first define the following elementary B-mappings on the set of positive real numbers $x \in R^+$:

(15) $B(p;x) = p+p/x$.

Next, given any sequence **s** of integers, let us define the following functions on R^+:

(16) $bf(m;x) = B(s_1;B(s_2;B(s_3;B(\dots B(s_m;x)\dots))))$, implying the recursion
$bf(1;x) \equiv B(s_1;x)$, $bf(m;x) = bf(m\text{-}1; B(s_m;x))$.

Clearly. if bf(m;x) converges for any x to the same value, then the Blazys' continued fraction bf(**s**) exists and equals the shared limit, i.e., $bf(\mathbf{s}) = \lim_{m\to\infty} bf(m;x)$. We will prove that, for any any $\mathbf{s} \in S$, this is indeed the case.

For any positive p, the mapping B(p;x) is a monotonously decreasing, continuous function of x and therefore also a bijection between positive real number intervals (a,b). Almost the same properties are shared by bf(m;x) which is an iterative nesting of B-mappings; the only difference is that bf(m;x) is monotonously *decreasing* for odd m and monotonously *increasing* for even m.

Given a sequence **s**, we will refer to intervals I_m, defined as the images of the full interval of $R^+ \equiv (0,\infty)$ under the mappings bf(m;x). From the monotonicity of B(p;x) and the iterative definition of bf(m;x) if follows that, for any natural index k, $I_{k+1} \subseteq I_k$ and therefore also $d_{k+1} \le d_k$, d_k being the size of I_k. Clearly, if we could show that all these nested intervals converge to a single point, i.e., that $\lim_{m\to\infty} d_m = 0$, the proof would be finished.

Explicit verification shows that a nested pair of the B- mappings, B(p;B(q;x)), maps the interval $(0,\infty)$ onto a finite interval $I_{pq} \equiv (p, p+p/q)$ which, for $p \le q$, is nested in (p, p+1). Furthermore, the pair B(p;B(q;x)) maps a finite interval (a,b) of size $d = b\text{-}a$ onto a finite interval $(p+p/(q+q/a), p+p/(q+q/b))$ with size $d' = d\,(p/q)/[(1+a)(1+b)]$.

Now, consider bf(m;x) of equation 16, assuming an *even* $m > 3$. The last two of its nested B-mappings, i.e., $B(s_{m-1};B(s_m;x))$, reduce the starting x-interval $(0,\infty)$ into a finite one (a_{m-1}, b_{m-1}) nested in $(s_{m-1}, s_{m-1}+1)$ whose size is at most 1. The next pair of B-mappings, $B(s_{m-3};B(s_{m-2};x))$, maps (a_{m-1}, b_{m-1}) onto (a_{m-3}, b_{m-3}) which is nested in $(s_{m-3}, s_{m-3}+1)$ and whose size is, according to the last paragraph,

$$
\begin{aligned}
(b_{m-3} - a_{m-3}) &= (b_{m-1} - a_{m-1})\,(s_{m-3}/s_{m-2}) / [(1+ a_{m-1})(1+ b_{m-1})]\\
&\le (b_{m-1} - a_{m-1})\,(s_{m-3}/s_{m-2}) / [(1+ s_{m-1})(1+ s_{m-1})] \;\le\; (b_{m-1} - a_{m-1})/4.
\end{aligned}
$$

The last passage uses the fact that, since the terms of **s** are positive and non-decreasing, $(s_{m-3}/s_{m-2}) \le 1$ and $s_{m-1} \ge 1$. Now the procedure can be iterated "backwards", terminating with the B-mappings pair $B(s_1;B(s_2;x))$. Since the size of the 'starting' interval does not exceed 1, and subsequent pairs of steps always reduce the size of the image interval by a factor of at least 4, we conclude that for even m, $d_{m+2} \le 2^{-m}$. For odd m>3 one can follow the same procedure but starting with three initial B-mappings; the final conclusion is that, for odd m, $d_{m+3} \le 2^{m}$. Hence, $\lim_{m\to\infty} d_m = 0$ and point (ii) is proved[6].

Proof of point (iii):

Let x and x' be two elements of X, differing by d = |x-x'| > 0. We want to show that the two integer sequences **s** = bx(x) and **s**' = bx(x') are not identical.

Set $x^{(1)} = x$, $x'^{(1)} = x'$, and $d^{(1)} = |x^{(1)} - x'^{(1)}| = d$, the superscript in parentheses denoting the iteration number. By definition, the first terms of **s** and **s**' are $s_1 = \text{floor}(x^{(1)})$ and $s'_1 = \text{floor}(x'^{(1)})$, respectively.

If s_1 and s'_1 differ, the proof is finished. Assume, therefore, that $s_1 = s'_1$ and proceed to next iteration.

The next terms of the sequences **s**, **s**' are generated in the same way, but starting with the modified numbers $x^{(2)} = 1/(x^{(1)}/s_1 - 1)$ and $x'^{(2)} = 1/(x'^{(1)}/s'_1 - 1)$, which differ by the amount $d^{(2)} = |x^{(2)} - x'^{(2)}| = s_1 d^{(1)}/((x^{(1)}-s_1)(x'^{(1)}-s'_1)) > s_1 d^{(1)}$. When $s_1 \ge 2$, the difference between the two numbers $c^{(k)}$ and $c'^{(k)}$ at least doubles in each iteration k, due to the fact that the sequence $s_1, s_2, s_3, \ldots$ is non-decreasing and therefore $s_k \ge 2 \Rightarrow d^{(k+1)} \ge 2d^{(k)}$ for any k. It therefore inevitably reaches a K such that $s_K = \text{floor}(x^{(K)})$ and $s'_K = \text{floor}(x'^{(K)})$ differ and the two sequences are distinct, which again terminates the proof.

There now remains the case of $s_1 = s'_1 = 1$ for which we have only $d^{(2)} > d^{(1)}$, a condition too weak to guarantee the divergence of the difference $d^{(k)}$. Note that in this case $1 < x^{(1)} < 2$ and $1 < x'^{(1)} < 2$. To analyse it further, let us proceed to yet another iteration, in which $s_2 = \text{floor}(x^{(2)})$ and $s'_2 = \text{floor}(x'^{(2)})$.

As before, if $s_2 \ne s'_2$, or if $s_2 = s'_2 \ge 2$, the proof is finished, so we can focus only on $s_2 = s'_2 = 1$. This implies $1 < x^{(2)} < 2$ and $1 < x'^{(2)} < 2$ which, combined with the analogous inequalities for $x^{(1)}$ and $x'^{(1)}$ (see above), gives $3/2 < x^{(1)} < 2$ and $3/2 < x'^{(1)} < 2$ or, equivalently, $0 < 2 - x^{(1)} < 0.5$ and $0 < 2 - x'^{(1)} < 0.5$.

According to the definition, the modified third-iteration x-values are in this case $x^{(3)} = 1/(x^{(2)} - 1)$ and $x'^{(3)} = 1/(x'^{(2)} - 1)$. Hence $d^{(3)} = |x^{(3)} - x'^{(3)}| = |1/(x^{(2)} - 1) - 1/(x'^{(2)} - 1)| = |1/(1/(x^{(1)} - 1) - 1) - 1/(1/(x'^{(1)} - 1) - 1)| = |x^{(1)} - x'^{(1)}|/((2 - x^{(1)})(2 - x'^{(1)})) = d^{(1)}/((2 - x^{(1)})(2 - x'^{(1)})) > 4d^{(1)}$.

This shows that, even in the most "stubborn" case, the difference between the two x-values increases at least by a factor of 4 every two iterations. It therefore diverges and, for some finite K, unavoidably leads to a difference between s_K and s'_K. With this, point (iii) is proved[7].

In practice the divergence of the difference |x-x'| is much faster than this lower bound of, on the average, a factor of 2 per iteration. Even in the slowest case of golden ratio, for example, the sequences for Φ and Φ+2^(-1000) differ already in 721-ebx term instead of 1000-th. Moreover, as we have seen, in a sequence containing a term of size $s_k > 2$ the divergence beyond k-th term is faster than s_n times per iteration. For example, in bx(π) the third term is 111, so that beyond k=3 the divergence is at least 111 times per iteration.

Proof of point (iv):

At this point, we need to show only that if **s** and **s**' are two distinct sequences in S, than their continued fractions bf(**s**) and bf(**s**') are distinct. Since $s_1 < \text{bf}(\mathbf{s}) < s_1+1$, this is evidently true if K = 1 because then the values bf(**s**) and bf(**s**') fall in two different, and disjoint, intervals.

[6] In addition to the proof, we have also gained a glimpse into the convergence rate: during its numeric evaluation, the precision of the approximated bf(**s**) increases by at least a factor of 2 upon every iteration.

[7] Incidentally, it also implies the following statement: Given two distinct irrational real numbers x and x', the leading terms of their Blazys' expansions **s** = bx(x) and **s**' = bx(x') can be identical for at most K terms, where K = $\log_2(|x-x'|) - 1$.

Hence, let K be the first index k for which s_K and s'_K are different, and K > 1. Denote as **u** and **u**' the sequences such that $u_k = s_{k+K-1}$ and $u'_k = s'_{k+K-1}$, which differ already in the first term. Then, following the notation of equation 16,

(17) bf(**s**) = bf(K-1;bf(**u**)) and bf(**s'**) = bf(K-1;bf(**u**')),

and bf(K-1;x) is the same mapping for both sequences. We already know that y = bf(**u**) and y' = bf(**u**') are distinct because **u** and **u**' differ in their first term. We also know that bf(K-1;x), being monotonous and continuous, is a bijection between R^+ and its image set. Hence, bf(K-1;y) and bf(K-1;y') are distinct, which proves point (iv).

VI. Extensions

Having established a novel bijection, one always wonders whether it might be possible to extend its scope. Two obvious ways that come to mind in this case are:

a) The set of all '*irrational real numbers greater than 1*' can be replaced by all '*positive irrational real numbers*' simply by adding 1 to every element, i.e., *prepend* to bx(x) an elementary bijection between its source set and another one.
b) The set of all '*non-decreasing sequences of natural numbers*' can be replaced by all '*infinite sequences of non-negative integers with no leading zeroes*', i.e., *append* to bx(x) a bijection (the inverse of the running sum operator, sum(**s**)) between its image set and another one.

With these options, one ends up with a bijection ebx(x), and its inverse ebf(**s**), between the set of all ***positive irrational real numbers***, and that of all ***sequences of non-negative integers with no leading zeros***;:

(18) $\text{ebx}(x) \equiv \text{sum}^{-1}(\text{bx}(x+1))$, and
$\text{ebf}(\mathbf{s}) \equiv \text{bf}(\text{sum}(\mathbf{s})) - 1 = \text{gf}(s'_1, s'_2, s'_3, \ldots | \, s'_1, s'_2, s'_3, \ldots)$, where $s'_k = \Sigma_{i=1,k}\, s_i$.

Here are some examples of ebx(x) ↔ ebf(**s**) pairs, using an underline to mark any infinite trailing cycles:

ebx(π) = {4, 24, 83, 2, 45, 2, 51, 5, 309, 1109, 87, 5342, 1708, 6306, 11091, 32279, … }
ebx(e) = {3, 1, 18, 12, 459, 41, 171, 2141, 20343, 295, 7363, 421, 916189, 1777526, … }
ebx(γ) = {1, 0, 0, 1, 0, 0, 0, 2, 8, 27, 32, 12, 401, 544, 419, 8466, 21629, 495338, 158789, …}
ebx(Φ) = {2, 1, 9, 4, $\underline{0}$}
ebx(√2) = {2, 2, $\underline{0}$}

ebf($\underline{1}$) = (3-e)/(e-2)
ebf(1,$\underline{0}$) = Φ-1 = 1/Φ
ebf($\underline{1,2}$) = 0.273704551020795232366760936554146207692896904906260676852 …
ebf(2,$\underline{0}$) = √3

Finally, the limitation to irrational real numbers could be also removed by admitting an infinite natural number ∞. Then, for example, the value 1.5 would generate the non-decreasing sequence

bx(3/2) = {1, 2, ∞, ∞, ∞, …} ≡ {1, 2, $\underline{\infty}$} with an inverse bf(1, 2, $\underline{\infty}$) = 3/2.

The result is a valid bijection between '*real numbers not smaller than 1*' and '*non-decreasing sequences of natural numbers, admitting also ∞*'. A problem is that it is not compatible with the above option (b) because, with the addition of the ∞ element, the running-sum operation is no longer a bijection. An alternative is to allow finite sequences, devise a way to encode them (for example by means of a conventional terminator, such as -1) and modify accordingly the iterative definitions.

However, it is not quite clear yet whether these efforts will ever 'pay off' in the sense of leading to any novel mathematical insights. For the time being, we just mention them.

VII. Concluding remarks

We have shown that there are bijections between

a) The set of *irrational real numbers greater than 1*,
b) The set of *non-decreasing sequences **s** of natural numbers*, and
c) A *subset of generalized continued fractions*, bf(**s**) = $s_1+s_1/(s_2+s_2/(s_3+s_3(\ldots)))$, with **s** belonging to (b).

This resembles closely the well-known bijections between

A. The set of *irrational real numbers greater than 1*,
B. The set of *sequences **n** of natural numbers*, and
C. The set of *simple continued fractions*, cf(**n**) = $n_1+1/(n_2+1/(n_3+\ldots))$, with **s** belonging[8] to (B).

Since simple continued fractions are also a subset of generalized continued functions, one cannot help but wonder how many *sets of bijections*, similar to the two listed above and based on different subsets of the generalized continued fractions, might exist.

The educational value of these bijections is one reason why the proofs in Section V were carried out in their entirety rather than referring certain parts to literature. It is a nice thing to dispose of multiple instances of bijections between the same set of '*irrational numbers greater than 1*' and various types of integer sequences, because it not only shows that the latter have the same cardinality (that is usually trivial), but it also enables the definition of explicit bijections between them.

For example, a bijection between the set of '*sequences of natural numbers*' (**n**) and its proper subset of '*non-decreasing sequences of natural numbers*' (**s**) can be defined as follows. Use (C) to convert **n** into an irrational x and then use bx(x) to expand the latter into **s**. To perform the inverse, instead, use (c) to convert **s** into an irrational x, and then expand the latter into a simple continued fraction to recover **n**.

As shown in the preceding Section, one can combine these recipes with other bijections between various sets of integer sequences, generating a staggering number of combinations. Two simple examples of such additional bijections are:

(i) the running sum mapping between '*sequences of non-negative integers with no leading zeroes*' and '*non-decreasing sequences of naturals*',
(ii) the running product between '*sequences of naturals*' and '*sequences of naturals divisible by their predecessor*' which is a subset of '*non-decreasing sequences of naturals*'. Exploring the possible bijections between these subsets can constitute nice class exercises.

A word about the linked OEIS entries:
I have registered twelve OEIS items: Five (A233582 to A233586) are Blazys' expansions of important math constants. One is the Blazys' expansion of √7 (the first √n with an aperiodic expansion). Four (from A233588 to A233591) are constants corresponding to a few non-decreasing sequences of natural numbers with different growth rates, and two (A233592 and A233593) list the natural numbers n for which bx(n) becomes periodic or a-periodic, respectively. One could, of course, proceed and generate numberless sequences and constants, but such an exercise appears to be pointless until some of these acquire a special prominence.

Note that, while the numeric evaluation of bf(**s**) is generally straightforward because of the fast convergence, the inverse expansion bx(x) usually requires x to be defined with a very high precision to

[8] For simple continued fractions, it is usual to allow n_1 to be 0. Here we avoid this case in order to make the correspondence between the two sets of bijections as close as possible.

compute, say, 1000 terms of the sequence. Precisions of the order of over 100'000 decimal digits are often called for and, in some cases, may be still insufficient.

An interesting difference between simple and Blazys' continued fractions regards the square roots of natural numbers (excluding squares). These have always periodic simple expansions [3], but with various cycle lengths. Since Blazys' continued fractions may not have a cycle length greater than 1, it is not surprizing that not all square roots of natural numbers become periodic.

What is surprising, perhaps, is that many do so and those appear to be exclusively of the general form $bx(\sqrt{n}) = \{s_1, s_2, \underline{m}\}$ with exactly two leading terms $s_1 \leq s_2 < m$. According to equations 7and 8, this implies that $\sqrt{n} = s_1+s_1/(s_2+2s_2/(m+\sqrt{(m(m+4))}))$. The existence of Diophantine solutions of this equation for small values of n, like $s_1 = s_2 = 1$, $m = 2$ for $n = 3$, does not look particularly surprising, but results such as $bx(\sqrt{843}) = \{29, 841, \underline{1682}\}$ are more so. From numeric tests, it looks like there is no upper limit on the values of n for which $bx(\sqrt{n})$ becomes periodic after 2 terms, while there are no cases with one, or with more than two, leading terms. These, however, are so far only unproved conjectures.

Another difference regards the n-th roots of the number e = exp(1) which appear to have periodic simple continued fractions [19], but their Blazys' continued fractions are a-periodic. This, again, appears to be due to the restriction to cycle length of 1. From this point of view, the extended mappings ebx(x) and ebf(**s**) of Section VI (which do admit cycles of any lengths) might yet come handy.

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